

# Integrability of the bi-Yang-Baxter $\sigma$ -model

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## **Abstract**

We construct a Lax pair with spectral parameter for a two-parameter doubly Poisson-Lie deformation of the principal chiral model.

# 1 Introduction

Let  $\mathcal{G}$  be a simple compact Lie algebra with its Killing-Cartan form  $(\cdot, \cdot)_{\mathcal{G}}$ . An  $\mathbb{R}$ -linear map  $R : \mathcal{G} \rightarrow \mathcal{G}$  is called a Yang-Baxter operator if it satisfies the skew-symmetry condition

$$(RX, Y)_{\mathcal{G}} + (X, RY)_{\mathcal{G}} = 0 \quad (1)$$

and the following variant of the modified Yang-Baxter equation

$$[RX, RY] = R([RX, Y] + [X, RY]) + [X, Y], \quad X, Y \in \mathcal{G}. \quad (2)$$

The canonical example of the Yang-Baxter operator is given by

$$RT^{\mu} = 0, \quad RB^{\alpha} = C^{\alpha}, \quad RC^{\alpha} = -B^{\alpha}, \quad (3)$$

where

$$T^{\mu} = iH^{\mu}, \quad B^{\alpha} = \frac{i}{\sqrt{2}}(E^{\alpha} + E^{-\alpha}), \quad C^{\alpha} = \frac{1}{\sqrt{2}}(E^{\alpha} - E^{-\alpha}), \quad (4)$$

and  $(H^{\mu}, E^{\alpha})$  is the usual Cartan-Weyl basis of the complex Lie algebra  $\mathcal{G}^{\mathbb{C}}$  (see Sec. 2.2. of [14] for more details).

Yang-Baxter operators play important role in the theory of integrable systems (cf. e.g. [17]), but it was only recently when they entered in the description of *geometries* of target spaces of certain integrable non-linear  $\sigma$ -models in  $1 + 1$ -dimensions [13, 14]. The integrable models in question were baptised the Yang-Baxter  $\sigma$ -models accordingly and, for a given simple compact group  $G$ , the action of such a model reads [13]

$$S_{\beta}(g) = \int_W (g^{-1} \partial_+ g, (I - \beta R)^{-1} g^{-1} \partial_- g)_{\mathcal{G}}. \quad (5)$$

Here  $I : \mathcal{G} \rightarrow \mathcal{G}$  is the identity map,  $\partial_{\pm}$  are the derivatives with respect to the light-cone coordinates  $\xi_{\pm}$  on the world-sheet  $W$  and  $g(\xi_+, \xi_-)$  is a group-valued field configuration. We observe that for  $\beta = 0$  we recover from (5) the action of the principal chiral model [22].

The Yang-Baxter  $\sigma$ -model (5) was subsequently reobtained by Delduc, Magro and Vicedo [4] in a new way, which made more transparent its integrability and also its symmetry structure. Indeed, though the  $\beta$ -deformation breaks the right  $G$ -symmetry of the principal chiral model (while the left

symmetry is kept intact), the right translations still continue to act as the so called Poisson-Lie symmetries [14]. The corresponding conserved Poisson-Lie charges were then explicitly computed in [4]. Delduc, Magro et Vicedo have also generalized the Yang-Baxter deformation to the coset target spaces and even to the supercoset target  $AdS_5 \times S^5$  important for the superstring theory [5]. Further superstring applications of this result followed readily [1, 12].

Is there a two-parameter deformation of the principal chiral model which would convert both left and right symmetries into the Poisson-Lie symmetries? The answer to this question is affirmative and the corresponding "bi-Yang-Baxter  $\sigma$ -model" was constructed in [14]. Its action reads

$$S_{\alpha,\beta}(g) = \int_W (g^{-1} \partial_+ g, (I - \alpha R_g - \beta R)^{-1} g^{-1} \partial_- g)_{\mathcal{G}}, \quad (6)$$

where  $R_g \equiv \text{Ad}_{g^{-1}} R \text{Ad}_g$ .

In this article, we establish the integrability of the bi-Yang-Baxter  $\sigma$ -model making it the first known integrable model on a group manifold having no apparent non-Abelian symmetries. We note in this respect, that the very few integrable  $\sigma$ -models on group targets known up to now are all symmetric with respect to an appropriate action of the group on which they live. This is the case for specific metric deformations of principal chiral model on the  $SU(2)$  group [3, 10, 11] and on the Nappi-Witten group [16], respectively, for the Yang-Baxter  $\sigma$ -model on every simple compact group [13, 14] and, finally, for the model interpolating between exact WZNW CFT and non-Abelian T-dual of principal chiral model on every simple compact group [2, 8, 20].

The lack of symmetry of the bi-Yang-Baxter  $\sigma$ -model makes difficult to search a suitable Lax pair starting from some ansatz, since it is not obvious how such ansatz should depend on the group element  $g$ . For this precise reason we left the problem open in [14] and decided to re-examine it only when the paper [4] appeared. We hoped to adapt the reasoning of Delduc, Magro and Vicedo to the presence of two deforming parameters  $\alpha$  and  $\beta$  but, quite unexpectedly, our reimmersion in the old stuff permitted us to identify the needed Lax pair by further developing the methods used already in [14]! We shall thus argue, that the bi-Yang-Baxter Lax pair has the following form

$$L_{\pm}^{\alpha,\beta}(\zeta) = \mp \left( \beta(R - i) + \frac{2i\beta \pm (1 + \alpha^2 - \beta^2)}{1 \pm \zeta} \right) (I \pm \alpha R_g \pm \beta R)^{-1} g^{-1} \partial_{\pm} g, \quad (7)$$

where  $\zeta$  is a complex valued spectral parameter. We note that for  $\alpha = \beta = 0$  the bi-Yang-Baxter  $\sigma$ -model (6) becomes the principal chiral model and (7)

becomes the standard Lax pair introduced by Zakharov and Mikhailov in [22]:

$$L_{\pm}^0(\zeta) = -\frac{g^{-1}\partial_{\pm}g}{1 \pm \zeta}. \quad (8)$$

The plan of the article is as follows: in Section 2, we first review the dynamics of the bi-Yang-Baxter  $\sigma$ -model, write its field equations and its Bianchi identities in terms of suitable currents and prove that the expression (7) gives indeed the bi-Yang-Baxter Lax pair. We then discuss some limiting cases of the parameters  $\alpha$  and  $\beta$  and, as a by-product of this discussion, we establish the gauge equivalence of the Lax pairs of the simple Yang-Baxter  $\sigma$ -model obtained in [14] and in [4], respectively. In Section 3, we discuss the notion of the extended solution of the Yang-Baxter  $\sigma$ -model and show how the bi-Yang-Baxter Lax pair can be actually *derived* from it, so that no guess-work was necessary to obtain the quite complicated expression (7). We finish by a short outlook.

## 2 Lax pair of the bi-Yang-Baxter $\sigma$ -model

We begin by noting that the relations (1) and (2) imply that the antisymmetric bracket

$$[X, Y]_R \equiv [RX, Y] + [X, RY], \quad X, Y \in \mathcal{G} \quad (9)$$

verifies the Jacobi identity and, hence, it defines a new Lie algebra structure  $\mathcal{G}_R \equiv (\mathcal{G}, [\cdot, \cdot]_R)$  on the vector space  $\mathcal{G}$ . We now introduce  $\mathcal{G}$ -valued "currents" by the prescription:

$$J_{\pm} := \mp(I \pm \alpha R_g \pm \beta R)^{-1} g^{-1} \partial_{\pm} g. \quad (10)$$

By varying the bi-Yang-Baxter action, we find that the corresponding field equations can be written as the following  $\mathcal{G}_R$  zero curvature condition:

$$\partial_+ J_- - \partial_- J_+ + \beta[J_-, J_+]_R = 0. \quad (11)$$

It follows from the definition (10) of the currents  $J_{\pm}$  that they fulfil certain Bianchi identities. They are given by:

**Lemma:** *For every solution  $g(\xi_+, \xi_-)$  of the field equations (10), (11) of the bi-Yang-Baxter model (6), the currents (10) verify also the following Bianchi identity:*

$$\partial_+ J_- + \partial_- J_+ + \beta[J_-, RJ_+] + \beta[J_+, RJ_-] + (1 + \alpha^2 - \beta^2)[J_-, J_+] = 0. \quad (12)$$

*Proof.* We start with the obvious Maurer-Cartan identity:

$$\partial_+(g^{-1}\partial_-g) - \partial_-(g^{-1}\partial_+g) - [g^{-1}\partial_-g, g^{-1}\partial_+g] = 0 \quad (13)$$

which, using the definition (10), can be rewritten as follows

$$\begin{aligned} & \partial_+((1 - \alpha R_g - \beta R)J_-) + \partial_-((1 + \alpha R_g + \beta R)J_+) + \\ & + [(1 - \alpha R_g - \beta R)J_-, (1 + \alpha R_g + \beta R)J_+] = 0. \end{aligned} \quad (14)$$

Using again (10), we now evaluate:

$$\begin{aligned} & \partial_+(R_g J_-) = \\ & = \partial_+(g^{-1}(R(gJ_-g^{-1}))g) = -[g^{-1}\partial_+g, R_g J_-] + R_g(\partial_+ J_-) + R_g[g^{-1}\partial_+g, J_-] = \\ & = [(1 + \alpha R_g + \beta R)J_+, R_g J_-] + R_g(\partial_+ J_-) - R_g[(1 + \alpha R_g + \beta R)J_+, J_-]. \end{aligned} \quad (15)$$

Similarly, we find

$$\begin{aligned} & \partial_-(R_g J_+) = \\ & = -[(1 - \alpha R_g - \beta R)J_-, R_g J_+] + R_g(\partial_- J_+) + R_g[(1 - \alpha R_g - \beta R)J_-, J_+]. \end{aligned} \quad (16)$$

Inserting (15) and (16) into (14), we obtain

$$\begin{aligned} & \partial_+ J_- + \partial_- J_+ + [J_-, J_+] + \beta[J_-, R J_+] + \beta[J_+, R J_-] \\ & - (\alpha R_g + \beta R)(\partial_+ J_- - \partial_- J_+ + \beta[J_-, J_+]_R) + \\ & + \beta^2 \left( [R J_+, R J_-] - R[R J_+, J_-] - R[J_+, R J_-] \right) \\ & - \alpha^2 \left( [R_g J_+, R_g J_-] - R_g[R_g J_+, J_-] - R_g[J_+, R_g J_-] \right) = 0. \end{aligned} \quad (17)$$

The second line of (17) vanishes because of the field equations (11) and the last two lines simplify because of the modified Yang-Baxter equation (2) (the modified Yang-Baxter equation is obviously verified also by the operator  $R_g$ ). We thus recover from (17) the desired Bianchi identity (12).  $\square$

In what follows, it will be convenient to introduce two current dependent expressions  $V_\pm$ :

$$V_\pm := \pm \partial_\pm J_\mp \pm \beta[J_\mp, R J_\pm] \pm \frac{1}{2}(1 + \alpha^2 - \beta^2)[J_-, J_+]. \quad (18)$$

Note that the field equations (11) and the Bianchi identities (12) can be cast, respectively, as

$$V_+ + V_- = 0, \quad V_+ - V_- = 0. \quad (19)$$

**Theorem (Lax pair):** *The following expressions give the Lax pair of the bi-Yang-Baxter  $\sigma$ -model:*

$$L_{\pm}^{\alpha,\beta}(\zeta) = \mp \left( \beta(R-i) + \frac{2i\beta \pm (1 + \alpha^2 - \beta^2)}{1 \pm \zeta} \right) (I \pm \alpha R_g \pm \beta R)^{-1} g^{-1} \partial_{\pm} g. \quad (20)$$

*This means, in other words, that for every solution  $g(\xi_+, \xi_-)$  of the field equations of the bi-Yang-Baxter model (6) and for a generic value of the complex spectral parameter  $\zeta$  the following zero-curvature condition holds:*

$$\partial_+ L_-^{\alpha,\beta}(\zeta) - \partial_- L_+^{\alpha,\beta}(\zeta) + [L_-^{\alpha,\beta}(\zeta), L_+^{\alpha,\beta}(\zeta)] = 0. \quad (21)$$

*Proof.* Using (10), we first rewrite

$$L_{\pm}^{\alpha,\beta}(\zeta) = \left( \beta(R-i) + \frac{2i\beta \pm (1 + \alpha^2 - \beta^2)}{1 \pm \zeta} \right) J_{\pm} \quad (22)$$

and then we use the modified Yang-Baxter equation (2) to calculate straightforwardly:

$$\begin{aligned} & \partial_+ L_-^{\alpha,\beta}(\zeta) - \partial_- L_+^{\alpha,\beta}(\zeta) + [L_-^{\alpha,\beta}(\zeta), L_+^{\alpha,\beta}(\zeta)] = \\ & = \left( \beta(R-i) + \frac{2i\beta + (1 + \alpha^2 - \beta^2)}{1 + \zeta} \right) V_- + \left( \beta(R-i) + \frac{2i\beta - (1 + \alpha^2 - \beta^2)}{1 - \zeta} \right) V_+. \end{aligned} \quad (23)$$

Eqs. (19) then imply that both  $V_{\pm}$  vanish when  $g(\xi_+, \xi_-)$  is a solution of the field equations of the bi-Yang-Baxter model (6).  $\square$

Let us now discuss the integrability of the bi-Yang-Baxter models for some special values of the parameters  $\alpha$  and  $\beta$ . We have already mentioned in the Introduction that for  $\alpha = \beta = 0$  the Lax pair (20) becomes the Zakharov-Mikhailov Lax pair (8). For  $\beta = 0$  the action (6) of the bi-Yang-Baxter model becomes

$$S_{\alpha}(g) = \int_W (g^{-1} \partial_+ g, (I - \alpha R_g)^{-1} g^{-1} \partial_- g)_{\mathcal{G}}. \quad (24)$$

This does not look like the Yang-Baxter  $\sigma$ -model (5), but, in fact, it coincides with it upon the transformation  $g \rightarrow g^{-1}$  and the change of the parameters

$\alpha \rightarrow \beta$ . The action of the Yang-Baxter  $\sigma$ -model was written in the form (24) in [6] where also the Lax pair of this model was written as

$$L_{\pm}^{\alpha}(\zeta) = -\frac{1+\alpha^2}{1\pm\zeta}(I \pm \alpha R_g)^{-1}g^{-1}\partial_{\pm}g. \quad (25)$$

We thus observe with satisfaction that our bi-Yang-Baxter Lax pair (20) gives for  $\beta = 0$  precisely the expression (25).

In the limit  $\alpha = 0$ , we obtain in turn

$$L_{\pm}^{\beta}(\zeta) = \left( \beta^2 \mp \beta R - \frac{1+\beta^2}{1\pm\lambda(\zeta)} \right) (I \pm \beta R)^{-1}g^{-1}\partial_{\pm}g, \quad (26)$$

where

$$\lambda(\zeta) = \frac{\zeta - i\beta}{1 - i\zeta\beta}. \quad (27)$$

We observe, that Eq. (26) gives nothing but the Lax pair of the Yang-Baxter  $\sigma$ -model (5) as given in [14].

Why two differently looking expressions (25) and (26) can be both Lax pairs of the same model? This happens because the Lax pair of an integrable model is not defined uniquely. Apart from the linear fractional transformations of the spectral parameter of the type (27), also any gauge transformation of the Lax connection gives a good Lax pair. Indeed, let us calculate the gauge transformation of (25), which obviously preserves the zero-curvature condition:

$$L_{\pm}^{\alpha}(\zeta) \rightarrow gL_{\pm}^{\alpha}(\zeta)g^{-1} + \partial_{\pm}gg^{-1} = -\left( \alpha^2 \mp \alpha R - \frac{1+\alpha^2}{1\pm\zeta^{-1}} \right) (I \pm \alpha R)^{-1}\partial_{\pm}gg^{-1}. \quad (28)$$

The transformation  $g \rightarrow g^{-1}$ ,  $\alpha \rightarrow \beta$  that transforms the version (24) of the Yang-Baxter  $\sigma$ -model into the version (5), transforms also (up to the inessential linear fractional transformation of the spectral parameter) the Lax pair (28) into the Lax pair (26). In this way we have proven the equivalence of the Lax pairs of the Yang-Baxter  $\sigma$ -model obtained in [14] and in [4, 6].

Finally, we comment the last special case  $\alpha = \beta$ . While the one-parameter Yang-Baxter deformation breaks the left-right symmetry of the principal chiral model (i.e. the symmetry with respect to the field transformation  $g \rightarrow g^{-1}$ ), the bi-Yang-Baxter model for  $\alpha = \beta$  does preserve this discrete symmetry. We expect that this fact will bring about new special properties of the model for  $\alpha = \beta$ .

### 3 Extended solutions

The present article would not be complete, if we did not explain, how we have found the bi-Yang-Baxter Lax pair (20). In fact, it was by no means the fruit of a guess work but rather the exploitation of properties of the so-called extended solutions of the integrable  $\sigma$ -models. The concept of the extended solution [21] plays an important role in the studies of the principal chiral model, in particular in connection with the so called dressing symmetries [22, 19, 15, 7, 18]. We shall first explain what is the extended solution of the principal chiral model and how to obtain the Lax pair of the Yang-Baxter  $\sigma$ -model out of it and then we shall consider the extended solution of the Yang-Baxter  $\sigma$ -model and obtain the Lax pair of the bi-Yang-Baxter  $\sigma$ -model out of it.

Let  $g_0 : W \rightarrow G$  be an ordinary solution of the principal chiral model and consider the associated Zakharov-Mikhailov Lax pair (8):

$$L_{\pm}^0(\zeta) = -\frac{g^{-1}\partial_{\pm}g}{1 \pm \zeta}. \quad (29)$$

Because  $g_0$  is solution, the zero curvature condition holds for generic  $\zeta$ :

$$\partial_+ L_-^0(\zeta) - \partial_- L_+^0(\zeta) + [L_-^0(\zeta), L_+^0(\zeta)] = 0 \quad (30)$$

hence there exists a map  $l_0(\zeta)$  from the (simply connected) world-sheet  $W$  to the complexified group  $G^{\mathbb{C}}$  such that

$$-l_0^{-1}(\zeta)\partial_{\pm}l_0(\zeta) = L_{\pm}^0(\zeta). \quad (31)$$

The map  $l_0(\zeta)$  is referred to as the *extended* solution associated to the ordinary solution  $g_0$ . Note also, that the ordinary solution can be extracted from the extended solution since it coincides with  $l_0(0)$  (possibly up to inessential left multiplication by a constant element from  $G$ ).

In the same token, let  $g_{\varepsilon} : W \rightarrow G$  be an ordinary solution of the Yang-Baxter  $\sigma$ -model (5) and consider the Lax pair (26) of this model obtained in [14]:

$$L_{\pm}^{\varepsilon}(\lambda) = \left( \varepsilon^2 \mp \varepsilon R - \frac{1 + \varepsilon^2}{1 \pm \lambda} \right) (I \pm \varepsilon R)^{-1} g^{-1} \partial_{\pm} g. \quad (32)$$

(Note that we have introduced the coupling constant  $\varepsilon$  instead of  $\beta$ ; the reason for this notation will become clear later). Because  $g_{\varepsilon}$  is solution, the



zero curvature condition holds for generic  $\lambda$ :

$$\partial_+ L_-^\varepsilon(\lambda) - \partial_- L_+^\varepsilon(\lambda) + [L_-^\varepsilon(\lambda), L_+^\varepsilon(\lambda)] = 0 \quad (33)$$

hence there exists a map  $l_\varepsilon(\lambda)$  from the (simply connected) world-sheet  $W$  to the complexified group  $G^\mathbb{C}$  such that

$$-l_\varepsilon^{-1}(\lambda)\partial_\pm l_\varepsilon(\lambda) = L_\pm^\varepsilon(\lambda). \quad (34)$$

The map  $l_\varepsilon(\lambda)$  will be referred to as the *extended* solution of the Yang-Baxter  $\sigma$ -model associated to the ordinary solution  $g_\varepsilon$ . Note also, that the ordinary solution  $g_\varepsilon$  can be extracted from the extended solution since it coincides with  $l_\varepsilon(\infty)$ .

In what follows, we shall also need the concept of the Iwasawa decomposition of the complexified group  $G^\mathbb{C}$  [23] in order to decompose in the Iwasawa way the extended solutions just discussed. To explain what is the Iwasawa decomposition, we first denote  $\mathcal{G}^\mathbb{C}$  the complexification of  $\mathcal{G}$  and view it as the real Lie algebra. Clearly, the multiplication by the imaginary unit  $i$  is  $\mathbb{R}$ -linear operator from  $\mathcal{G}^\mathbb{C} \rightarrow \mathcal{G}^\mathbb{C}$ , so its restriction on  $\mathcal{G}$  is well-defined  $\mathbb{R}$ -linear operator with the domain  $\mathcal{G}$  and the range  $\mathcal{G}^\mathbb{C}$ . Thus  $(R - i)$  can be also understood as a  $\mathbb{R}$ -linear operator from  $\mathcal{G}$  to  $\mathcal{G}^\mathbb{C}$ . Using the modified Yang-Baxter equation (2), it can be easily verified that  $(R - i)$  is in fact an injective homomorphism between the real Lie algebras  $\mathcal{G}_R$  and  $\mathcal{G}^\mathbb{C}$  and it thus permits to view  $\mathcal{G}_R$  as the real subalgebra of  $\mathcal{G}^\mathbb{C}$ . The subgroup  $G_R$  of  $G^\mathbb{C}$ , obtained by integrating the Lie subalgebra  $\mathcal{G}_R$  of  $\mathcal{G}^\mathbb{C}$ , turns out to be nothing but the so called group  $AN$ . Recall, that an element  $b$  of  $AN$  can be uniquely represented by means of the exponential map as follows

$$b = e^\phi \exp[\sum_{\alpha > 0} v_\alpha E^\alpha] \equiv e^\phi n.$$

Here  $\alpha$ 's denote the roots of  $\mathcal{G}^\mathbb{C}$ ,  $v_\alpha$  are complex numbers,  $E^\alpha$  are the step generators and  $\phi$  is an Hermitian element of the Cartan subalgebra of  $\mathcal{G}^\mathbb{C}$ . In particular, if  $G^\mathbb{C} = SL(n, \mathbb{C})$ , the group  $AN$  can be identified with the group of upper triangular matrices of determinant 1 and with positive real numbers on the diagonal.

### 3.1 Yang-Baxter Lax pair from principal chiral model

The Iwasawa theorem [23] guarantees the existence and the uniqueness of the map  $\text{Iw}: G^\mathbb{C} \rightarrow G$  such that, for every  $l \in G^\mathbb{C}$ , the product  $\text{Iw}(l)l^{-1}$  belongs

to  $AN$ . Consider now an extended solution  $l_0(\zeta)$  of the principal chiral model associated to some ordinary solution  $g_0$  and define

$$g_\varepsilon := \text{Iw}(l_0(-i\varepsilon)), \quad b_\varepsilon := l_0(-i\varepsilon)g_\varepsilon^{-1}. \quad (35)$$

We have therefore

$$\frac{g_0^{-1}\partial_\pm g_0}{1 \mp i\varepsilon} = l_0(-i\varepsilon)^{-1}\partial_\pm l_0(-i\varepsilon) = g_\varepsilon^{-1}b_\varepsilon^{-1}\partial_\pm b_\varepsilon g_\varepsilon + g_\varepsilon^{-1}\partial_\pm g_\varepsilon. \quad (36)$$

Because  $b_\varepsilon^{-1}\partial_\pm b_\varepsilon \in \text{Lie}(AN)$ , there exists  $K_\pm(\xi_+, \xi_-) \in \mathcal{G}$  such that

$$b_\varepsilon^{-1}\partial_\pm b_\varepsilon = \varepsilon(R - i)K_\pm. \quad (37)$$

This fact and Eq.(36) permit to infer that

$$g_0^{-1}\partial_\pm g_0 = (1 \mp i\varepsilon)(\varepsilon g_\varepsilon^{-1}(R - i)K_\pm g_\varepsilon + g_\varepsilon^{-1}\partial_\pm g_\varepsilon). \quad (38)$$

Note that the left hand side of Eq.(38) is  $\mathcal{G}$ -valued but the right hand side is  $\mathcal{G}^{\mathbb{C}}$ -valued. This is possible only if the  $i\mathcal{G}$ -part of the right hand side vanishes:

$$i\varepsilon g_\varepsilon^{-1}(K_\pm \pm \varepsilon R K_\pm \pm \partial_\pm g_\varepsilon g_\varepsilon^{-1})g_\varepsilon = 0$$

that is

$$K_\pm = \mp(1 \pm \varepsilon R)^{-1}\partial_\pm g_\varepsilon g_\varepsilon^{-1}. \quad (39)$$

By inserting (39) back into (38), we obtain

$$g_0^{-1}\partial_\pm g_0 = \frac{1 + \varepsilon^2}{I \pm \varepsilon R_{g_\varepsilon}} g_\varepsilon^{-1}\partial_\pm g_\varepsilon, \quad (40)$$

or

$$-\frac{g_0^{-1}\partial_\pm g_0}{1 \pm \zeta} = -\frac{1 + \varepsilon^2}{1 \pm \zeta}(I \pm \varepsilon R_{g_\varepsilon})^{-1}g_\varepsilon^{-1}\partial_\pm g_\varepsilon. \quad (41)$$

Note that on the left hand side of (41) is the Lax pair (8) of the principal chiral model and on the right hand side is the Lax pair (25) of the Yang-Baxter  $\sigma$ -model in the version of [6].

We remark, that we were not aware about this simple method to obtain the Lax pair of the Yang-Baxter  $\sigma$ -model when we had been writing the paper [14]. As a matter of fact, however, we are benefiting from this circumstance here. Why? Because the alternative version (26) of the Yang-Baxter Lax pair, that we have obtained in [14] in a more complicated way, serves in this article as the starting point for obtaining the bi-Yang-Baxter Lax pair by the simple method presented in this subsection. Surprisingly enough, the extended solution obtained from the other Lax pair (25) turns out to be of no direct utility in this respect, as it can be easily checked.

### 3.2 Bi-Yang-Baxter Lax pair from Yang-Baxter model

Consider now the extended solution  $l_\varepsilon(\lambda)$  of the Yang-Baxter  $\sigma$ -model associated to some ordinary solution  $g_\varepsilon$ . Recall that  $l_\varepsilon(\lambda)$  verifies (34). For a real  $\eta$ , we define

$$g_{\varepsilon\eta} := \text{Iw}(l_\varepsilon(-i\eta)), \quad b_{\varepsilon\eta} = l_\varepsilon(-i\eta)g_{\varepsilon\eta}^{-1}. \quad (42)$$

We have therefore

$$\begin{aligned} & -\left(\varepsilon^2 \mp \varepsilon R - \frac{1 + \varepsilon^2}{1 \mp i\eta}\right)(I \pm \varepsilon R)^{-1}g_\varepsilon^{-1}\partial_\pm g_\varepsilon = l_\varepsilon(-i\eta)^{-1}\partial_\pm l_\varepsilon(-i\eta) = \\ & = g_{\varepsilon\eta}^{-1}b_{\varepsilon\eta}^{-1}\partial_\pm b_{\varepsilon\eta}g_{\varepsilon\eta} + g_{\varepsilon\eta}^{-1}\partial_\pm g_{\varepsilon\eta} = g_{\varepsilon\eta}^{-1}\eta(R - i)K_\pm g_{\varepsilon\eta} + g_{\varepsilon\eta}^{-1}\partial_\pm g_{\varepsilon\eta}. \end{aligned} \quad (43)$$

Indeed, because of  $b_{\varepsilon\eta}^{-1}\partial_\pm b_{\varepsilon\eta} \in \text{Lie}(AN)$ , there exists  $K_\pm(\xi_+, \xi_-) \in \mathcal{G}$  such that

$$b_{\varepsilon\eta}^{-1}\partial_\pm b_{\varepsilon\eta} = \eta(R - i)K_\pm. \quad (44)$$

In analogy with the previous subsection, we know infer from Eq.(43) that the following expression must have the vanishing  $i\mathcal{G}$ -part :

$$\left(\varepsilon^2 \mp \varepsilon R - \frac{1 + \varepsilon^2}{1 \pm i\eta}\right)\left(g_{\varepsilon\eta}^{-1}\eta(R - i)K_\pm g_{\varepsilon\eta} + g_{\varepsilon\eta}^{-1}\partial_\pm g_{\varepsilon\eta}\right). \quad (45)$$

This condition permits to determine  $K_\pm$ :

$$K_\pm = \mp \frac{1 + \varepsilon^2}{1 - \varepsilon^2\eta^2} \left( I \pm \frac{\eta(1 + \varepsilon^2)}{1 - \varepsilon^2\eta^2} R \pm \frac{\varepsilon(1 + \eta^2)}{1 - \varepsilon^2\eta^2} R_{g_{\varepsilon\eta}^{-1}} \right)^{-1} \partial_\pm g_{\varepsilon\eta} g_{\varepsilon\eta}^{-1}. \quad (46)$$

By inserting (46) back into (43), we obtain after some calculation

$$(I \pm \varepsilon R)^{-1}g_\varepsilon^{-1}\partial_\pm g_\varepsilon = \frac{1 + \eta^2}{1 - \varepsilon^2\eta^2} \left( I \pm \frac{\eta(1 + \varepsilon^2)}{1 - \varepsilon^2\eta^2} R_{g_{\varepsilon\eta}} \pm \frac{\varepsilon(1 + \eta^2)}{1 - \varepsilon^2\eta^2} R \right)^{-1} g_{\varepsilon\eta}^{-1}\partial_\pm g_{\varepsilon\eta} \quad (47)$$

hence

$$\begin{aligned} & \left(\varepsilon^2 \mp \varepsilon R - \frac{1 + \varepsilon^2}{1 \pm \lambda}\right)(I \pm \varepsilon R)^{-1}g_\varepsilon^{-1}\partial_\pm g_\varepsilon = \\ & = \left(\varepsilon^2 \mp \varepsilon R - \frac{1 + \varepsilon^2}{1 \pm \lambda}\right) \frac{1 + \eta^2}{1 - \varepsilon^2\eta^2} \left( I \pm \frac{\eta(1 + \varepsilon^2)}{1 - \varepsilon^2\eta^2} R_{g_{\varepsilon\eta}} \pm \frac{\varepsilon(1 + \eta^2)}{1 - \varepsilon^2\eta^2} R \right)^{-1} g_{\varepsilon\eta}^{-1}\partial_\pm g_{\varepsilon\eta} = \end{aligned}$$

$$= \mp \left( \beta(R - i) + \frac{2i\beta \pm (1 + \alpha^2 - \beta^2)}{1 \pm \zeta} \right) (I \pm \alpha R_{g_{\varepsilon\eta}} \pm \beta R)^{-1} g_{\varepsilon\eta}^{-1} \partial_{\pm} g_{\varepsilon\eta}, \quad (48)$$

where

$$\zeta = \frac{\lambda + i\varepsilon}{1 + i\varepsilon\lambda}, \quad \alpha \equiv \frac{\eta(1 + \varepsilon^2)}{1 - \varepsilon^2\eta^2}, \quad \beta \equiv \frac{\varepsilon(1 + \eta^2)}{1 - \varepsilon^2\eta^2}. \quad (49)$$

Note that on the extreme left hand side of (48) is the Lax pair (26) of the Yang-Baxter  $\sigma$ -model and on the extreme right hand side is the Lax pair (7) of the bi-Yang-Baxter  $\sigma$ -model.

## 4 Conclusions and outlook

The principal result of this paper is the construction of the Lax pair (7) of the bi-Yang-Baxter  $\sigma$ -model. As far as the directions to develop further our work, we think that a promising one should consist in the study of the T-duality story which is naturally associated to any Poisson-Lie symmetric  $\sigma$ -model [9]. The bi-Yang-Baxter model is doubly Poisson-Lie symmetric and hence Poisson-Lie T-dualizable from both right and left side. Combining the left and the right T-duality, a novel nontrivial dynamical equivalence of two seemingly different  $\sigma$ -models living on the target of the dual Poisson-Lie group  $AN$  should be thus obtained. Another natural open problem is to find out whether the bi-Yang-Baxter Lax pair constructed in the present work can be obtained by a suitable adaptation of the method of Ref.[4].

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